

EXTENSION AND REDUCTION THEOREMS FOR CERTAIN TYPES
OF CONTINUOUS TRANSFORMATIONS

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EXTENSION AND REDUCTION THEOREMS FOR CERTAIN TYPES OF CONTINUOUS TRANSFORMATIONS.*

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1. Introduction. In recent years several types of single-valued continuous transformations have been defined and investigated by various authors. Of these transformations, some of the most interesting are the interior transformations defined by S. Stoilow [1]¹ the non-alternating transformations defined by G. T. Whyburn [2], the non-separating transformations of J. F. Wardwell [3], and the monotone transformations studied by Whyburn [2] and others. Stronger and weaker forms of the non-alternating and non-separating transformations have been considered by E. P. Vance [4]. These transformations will be defined as they are used in later sections of this paper.

In this paper we study what might be called "hereditary" properties of such transformations. That is, if P is one of the properties mentioned above (interiority, monotoneity, etc.), we inquire under what circumstances a transformation having property P on a space A will have property P on every subcontinuum of A , or on certain special subcontinua (arc-sets and cyclic elements) of A . The conditions for this "inheritance" of a property by a subcontinuum or set of subcontinua may take the form of conditions on the transformations or of conditions on the subcontinua. In addition to these reduction theorems, we give an extension theorem valid for several types of transformation.

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Unless otherwise specified, A will denote, throughout this paper, a Peano space (= continuous image of an interval = compact locally connected continuum), and T a single-valued continuous transformation. If a transformation $T(A) = B$ has property P , and if R is a subcontinuum of A , T will be said to *have property P on R* if $T(R) = S$, where S is a subset of R , has property P . Definitions of cyclic elements, arc-sets, etc. may be found in a paper by Kuratowski and Whyburn [5]; we assume that the reader is familiar with the definitions and results of this paper.

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¹ Figures in square brackets refer to the bibliography at the end of the paper.

2. Transformations possessing a property hereditarily. In this section we obtain necessary and sufficient conditions on transformations of certain types on a space A in order that these transformations shall possess their defining properties on every subcontinuum of A .

Our first theorem concerns monotone transformations, *i. e.*, transformations under which the inverse image of each point of the transform is connected.

THEOREM 2.1. *A necessary and sufficient condition that a transformation $T(A) = B$ be monotone on every subcontinuum of A is that the inverse image of each point of B be an arc-set.²*

This theorem follows readily from results of Whyburn [6, pp. 311-312].

A transformation is said to be *non-alternating* on A if, given any two points of the transform, neither inverse image separates the other in A . (Throughout this paper we use "separate" in the strong sense; *i. e.*, a set X is said to separate a set M in A if $A - X$ is the sum of two disjoint sets, each closed relative to $A - X$, and each containing at least one point of M .)

THEOREM 2.2. *A transformation $T(A) = B$ is monotone on every subcontinuum of A if and only if it is non-alternating on every subcontinuum of A .*

Since every monotone transformation is non-alternating, the necessity is immediate.

To prove the sufficiency, we use the characterization of transformations monotone on every subcontinuum of A given by the preceding theorem. Let y be a point of B , and suppose that $T^{-1}(y)$ is not an arc-set. Then there is an arc α in A with only its endpoints in $T^{-1}(y)$. But T is obviously not non-alternating on α , contrary to the hypothesis that T is non-alternating on every subcontinuum of A . Hence every $T^{-1}(y)$ must be an arc-set, and T must, therefore, be monotone on every subcontinuum of A .

We now obtain a necessary and sufficient condition, of different character from the condition of Theorem 2.1, in order that a transformation be monotone (or, by the preceding theorem, non-alternating) on every subcontinuum of the space. We shall say that a transformation is *constant* on a set E if it carries E into a single point.

THEOREM 2.3. *A necessary and sufficient condition that a monotone transformation $T(A) = B$ be monotone on every subcontinuum of A is that T be homeomorphic or constant on each cyclic element of A .*

² We use the term *arc-set* in the sense of *closed arc-set* (= *A-set* in the terminology of [6]).

The condition is necessary. Obviously T is constant on each degenerate cyclic element (= a single point). Now let C be a true cyclic element of A . If T is not homeomorphic on C , then, for some point y of $T(C)$, $T^{-1}(y) \cdot C$ is non-degenerate; and if T is not constant on C , there exists a point x of $T(C)$ distinct from y . Let p and r be distinct points of $T^{-1}(y) \cdot C$, and let q be a point of $T^{-1}(x) \cdot C$. By Ayres' Three-Point Theorem [7, p. 130, Th. 3], there exists an arc pqr in C . But, contrary to the hypothesis that T is monotone, and hence non-alternating, on every subcontinuum of A , T then fails to be non-alternating on the subcontinuum pqr , for $T^{-1}(y)$ intersects two components of $pqr - T^{-1}(x)$, one in p and the other in r .

The condition is sufficient.³ Let T be monotone on A and homeomorphic or constant on each cyclic element of A . Suppose that there is a continuum K in A and a point y of B such that $K \cdot T^{-1}(y) = M + N$, where the sets M and N are disjoint and closed. There is an arc α in A , with endpoints m and n , such that $\alpha \cdot M = m$ and $\alpha \cdot N = n$. Since K and $T^{-1}(y)$ are continua, it is easy to see that no point separates m from n in A , whence $m + n$ lies in a cyclic element E . Since T is thus not homeomorphic on E , we must have $E \subset T^{-1}(y)$. Now EK is connected, but $EK = E \cdot K \cdot T^{-1}(y) = EM + EN$, and the summands are mutually separated. This is a contradiction.

For the necessity part of Theorem 2.3, the condition that T be monotone on A is evidently superfluous. However, without this condition the sufficiency part of the theorem is not true. For example, let A be an arc xy and let T be a transformation which identifies x and y but is homeomorphic elsewhere on A .

As an immediate consequence of Theorems 2.2 and 2.3 we have

THEOREM 2.4. *A necessary and sufficient condition that a non-alternating transformation $T(A) = B$ be non-alternating on every subcontinuum of A is that T be homeomorphic or constant on each cyclic element of A .*

COROLLARY. *A necessary and sufficient condition that every non-alternating transformation on a Peano continuum A be non-alternating on every subcontinuum of A is that A be a dendrite.*

The sufficiency of the condition follows at once from Theorem 2.4, for every single-valued transformation on a dendrite is necessarily constant on every cyclic element. Conversely, on a non-acyclic Peano continuum we can always define a non-alternating transformation which, on some cyclic element, is neither homeomorphic nor constant, and which in consequence, by the above theorem, must fail to be non-alternating on some subcontinuum. Such a

³ The author is indebted to the referee for this abbreviated proof of the sufficiency.

transformation may be defined as one which carries into a point an arc in some true cyclic element of the continuum, and which is a homeomorphism elsewhere; this transformation is obviously monotone, and hence non-alternating.

A transformation $T(A) = B$ is said to be *weakly non-alternating* provided, for any two distinct points x and y of B , $T^{-1}(x)$ separates $T^{-1}(y)$ in A only if some single point of $T^{-1}(x)$ separates $T^{-1}(y)$ in A . The transformation would be called *completely non-alternating* if, for any two points x and y (not necessarily distinct) of B , and any closed subset K of $T^{-1}(x)$, K did not separate $T^{-1}(y)$ in A .

The necessity part of Theorem 2.4 fails if the hypothesis that T be non-alternating on every subcontinuum of A is replaced by the weaker hypothesis that T be weakly non-alternating on every subcontinuum of A . For example, let A be a circle and let T identify two points on the circle (T being a homeomorphism elsewhere on A); T is then weakly non-alternating on every subcontinuum of A (and, in fact, completely non-alternating on A), but is neither homeomorphic nor constant on the cyclic element A .

It will follow as a trivial corollary of Theorem 2.5 that the sufficiency part of Theorem 2.4 holds if "weakly non-alternating" replaces "non-alternating" throughout the statement of the theorem. However, if only the first occurrence of "non-alternating" is replaced by "weakly non-alternating," the theorem is no longer true. For example, consider the transformation which folds a line segment about its midpoint; this transformation is weakly non-alternating on the segment and is constant on every cyclic element, but is not non-alternating even on the whole segment. Since every transformation completely non-alternating on every subcontinuum of a Peano space is obviously a homeomorphism, it is apparent that replacement of the second occurrence of "non-alternating" by "completely non-alternating" will render the sufficiency theorem false unless the first occurrence of "non-alternating" is similarly replaced, in which case the theorem is trivially true.

A transformation $T(A) = B$ is called *non-separating* if the inverse image of no point of B separates A . It is *weakly non-separating* if an inverse image can separate A only if one of its points separates A . It is *completely non-separating* if no closed subset of any inverse image separates A .

Results similar to the above for the various types of non-separating transformations, insofar as they are not contained in the foregoing remarks, are to be found in Theorems 2.5, 2.6, and 2.7. There is one exception, however: it is impossible to define a transformation completely non-separating on every subcontinuum of a Peano space.

As a preliminary to the proof of Theorem 2.5 we prove the following

LEMMA. If H is a subcontinuum of a Peano space A , and K is a closed set which separates H between two points p and q , then there exists a cyclic element C of A such that KC separates H between p and q .

Consider the cyclic chain $C(p, q)$ from p to q . Since every cut point of $C(p, q)$ is a cut point of A , and hence a cyclic element, if K contains a cut point of $C(p, q)$ the lemma is proved. Let us suppose, then, that $K \cdot Z = 0$, where Z is the set of all cut points of $C(p, q)$. Since $H \cdot C(p, q)$ is a subcontinuum of $C(p, q)$ containing $p + q$, $Z \subset H \cdot C(p, q)$. But, by hypothesis, $H \cdot C(p, q) - K = H_p + H_q$, where H_p and H_q are mutually separated sets containing p and q , respectively. Hence $Z = Z_p + Z_q$, where Z_p and Z_q are mutually separated and contain p and q , respectively. Since $Z + p + q$ is closed, Z_p and Z_q are closed. The cyclic elements in $C(p, q)$ being linearly ordered between p and q , let p' be the last point of Z_p in the order from p to q , and let q' be the first point of Z_q in the order from p' to q . Since $p' \neq q'$, p' and q' must be the limit points in a certain cyclic element C of the components of $C(p, q) - C$ containing p and q , respectively. The points p' and q' being separated in H by K , they must then be separated by KC , whence p and q are separated in H by KC .

THEOREM 2.5. If $T(A) = B$ is homeomorphic or constant on each cyclic element of A , then T is weakly non-separating on every subcontinuum of A .

Suppose that there exists a subcontinuum H of A on which T is not weakly non-separating; i. e., suppose that there exists a point x of $T(H)$ such that $T^{-1}(x)$ separates H while no single point of $T^{-1}(x)$ separates H . By the foregoing lemma, there exists a cyclic element C of A such that $C \cdot T^{-1}(x)$ separates H . Now T cannot be homeomorphic on C , for if it were then $C \cdot T^{-1}(x)$ would be a single point, and we have supposed that no single point of $T^{-1}(x)$ separates H . On the other hand, T cannot be constant on C ; for in this case C would separate H , and, C being a cyclic element, C would contain a cut point of A which would separate H , contrary to supposition. But, by hypothesis, T is either homeomorphic or constant on C . Hence, by contradiction, T is weakly non-separating on every subcontinuum of A .

The converse of this theorem is false; for if the space A be taken to be a circle, any transformation carrying a semicircle into a single point, and homeomorphic elsewhere on A , is weakly non-separating on every subcontinuum of A but is neither homeomorphic nor constant on the cyclic element A . However, the hypothesis that T is weakly non-separating on every subcontinuum of A does yield the conclusion embodied in

THEOREM 2.6. *If $T(A) = B$ is weakly non-separating on every subcontinuum of A , then T is monotone on each cyclic element of A (in fact, $T^{-1}(x) \cdot E$ is arcwise connected for every point x of B and every cyclic element E of A).*

Let x be any point of B and E any cyclic element of A such that $T^{-1}(x) \cdot E \neq \emptyset$. If $T^{-1}(x) \cdot E$ is a single point, our conclusion follows trivially. Hence we suppose $T^{-1}(x) \cdot E$ to be non-degenerate and let p and q be distinct points of $T^{-1}(x) \cdot E$. Since $p + q$ lies in a true cyclic element E , there exists a simple closed curve J in E containing $p + q$; and J is the sum of two arcs, α_1 and α_2 , having only $p + q$ in common. Since T is, by hypothesis, weakly non-separating on J , and since no single point can separate a simple closed curve, it follows that $J - T^{-1}(x)$ is connected. Hence $T^{-1}(x)$ contains one of the arcs α_i from p to q , whence $T^{-1}(x) \cdot E$ is arcwise connected.

That the converse of this theorem is false is easily seen by considering a θ -curve A (i. e., two points, p and q , and three arcs which join p to q and each pair of which have in common only $p + q$) and a transformation which carries one of the arcs pq into a single point and is homeomorphic elsewhere on A . Each inverse image is arcwise connected, but the transformation is not weakly non-separating even on A .

THEOREM 2.7. *A necessary and sufficient condition that a non-separating transformation $T(A) = B$ be non-separating on every subcontinuum of A is that B be a single point.*

The sufficiency is obvious. To prove the necessity of the condition, suppose that there exist distinct points x and y of B . Let p be a point of $T^{-1}(x)$ and q a point of $T^{-1}(y)$. Since A is a Peano space, there exists an arc pq in A ; and since $T^{-1}(x)$ and $T^{-1}(y)$ are disjoint and closed, $pq - T^{-1}(x) - T^{-1}(y) \neq \emptyset$. Let t be a point of $pq - T^{-1}(x) - T^{-1}(y)$; then t is an interior point, and hence a cut point, of pq . But t is a point of $T^{-1}(z)$ for some point z of B ; and since $t \cdot T^{-1}(x) = t \cdot T^{-1}(y) = \emptyset$ we have $x \neq z \neq y$. Hence $p \cdot T^{-1}(z) = q \cdot T^{-1}(z) = \emptyset$, so that $T^{-1}(z)$ cuts pq , a subcontinuum of A , contrary to hypothesis.

THEOREM 2.8. *A necessary and sufficient condition that an interior transformation $T(A) = B$ be interior on every subcontinuum of A is that T be homeomorphic or constant on A .*

The sufficiency of the condition is obvious. To prove the necessity, suppose that B is not a single point but that there exist points x_1 and x_2 of A such that $T(x_1) = T(x_2)$. Since A is a Peano space, there exists at least one

are in A joining x_1 and x_2 ; let α be such an arc. We consider separately the mutually exclusive cases in which (1) there exists a point u of α such that $T(x_1u) \neq T(x_2u)$, where x_iu denotes the subarc of α joining x_i and u ($i = 1, 2$), and (2) $T(x_1u) = T(x_2u)$, for all points u of α .

Case 1. In this case, since $T(x_1u) \neq T(x_2u)$, there exists a point s of x_1u such that $T(s)$ is not a point of $T(x_2u)$ (or else there exists a point of x_2u whose transform does not belong to $T(x_1u)$; let us suppose the former). Now let p_1 be the first point on α , in the order from s to x_1 , such that $T(p_1)$ is a point of $T(x_2u)$ (there exists such a point, for $T^{-1}T(x_2u)$ is closed and at least x_1 itself satisfies the condition). Let p_2 be any point of x_2u such that $T(p_2) = T(p_1)$, let p_1s be the subarc of α from p_1 to s , let $U = T(p_1s - s)$, and let A_1 be the subarc p_1ux_2 of α . Then $p_1s - s$ is a set open relative to A_1 . Let $T(A_1) = B_1$. We proceed to show that $T(p_1s - s)$ is not open in B_1 , whence the hypothesis that T is interior on every subcontinuum of A is contradicted.

Since $T(p_1) = T(p_2)$, $T(p_1) \subset U \cdot T(x_2u)$; and since p_1 is the only point common to $p_1s - s$ and $T^{-1}T(x_2u)$, $U \cdot T(x_2u) \subset T(p_1)$. Hence $U \cdot T(x_2u) = T(p_1)$. Therefore, since $x_2u \subset A_1$, $T(x_2u) - T(p_1) \subset B_1 - U$. But p_2 is a limit point of a subset M of x_2u , whence $T(p_2)$ is, by the continuity of T , a limit point of $T(M)$, and $T(M) \subset T(x_2u) - T(p_1) \subset B_1 - U$. Therefore, since $T(p_1) = T(p_2)$, the point $T(p_1)$ of U is a limit point of $B_1 - U$, whence U is not open relative to B_1 .

Case 2. In this case we suppose that $T(x_1u) = T(x_2u)$ for every point u of the arc α joining x_1 and x_2 . If $T(\alpha)$ is a single point, T is obviously interior on α . But we are supposing that B is not a single point, so that there exists a point y in A such that $T(y) \neq T(\alpha)$. Let β be an arc joining α to y . Then T is not interior on the subcontinuum $\alpha + \beta$, contrary to hypothesis.

On the other hand, if we suppose that $T(\alpha)$ is not a single point, i. e., if we suppose there exists a point v of α such that $T(v) \neq T(x_1) = T(x_2)$, then $\rho(T(v), T(x_1)) = \epsilon$, where $\rho(T(v), T(x_1))$ is the distance between $T(v)$ and $T(x_1)$ and ϵ is a positive number. Since T is a continuous transformation, there exists a point u of α so close to x_1 that $\text{diam } T(x_1u) < \epsilon$, whence $T(x_1u)$ cannot contain $T(v)$. Hence v is a point of x_2u and $T(x_1u) \neq T(x_2u)$, contrary to supposition.

3. Conditions on the subcontinua. In this section we obtain necessary and sufficient conditions on a subcontinuum R of a Peano space A in order that every transformation having one of the aforementioned properties on A shall have the same property on R . In order to state the theorems in the strongest

possible form, we shall in some cases state separately the necessary and the sufficient conditions.

THEOREM 3.1. *In order that every monotone transformation $T(A) = B$ be monotone on a given subcontinuum R of A , it is necessary and sufficient that R be an arc-set.*

The condition is necessary. If R is not an arc-set, then, for some pair of points x and y in R , there exists an arc xy not lying wholly in R . Let T be a transformation carrying xy into a single point p and homeomorphic on $A - xy$. Then T is monotone on A but not on R ; for an arc xy is irreducibly connected between x and y , whence $xy \cdot R (= T^{-1}(p) \cdot R)$ is not connected.

The condition is sufficient. Since T is monotone on A , $T^{-1}(x)$ is connected for every point x of B . But, since R is an arc-set, RC is connected for every connected set C in A . Hence $T^{-1}(x) \cdot R$ is connected for every point x of B , and T is therefore monotone on R .

In view of the obvious fact that a Peano space is a dendrite if and only if all its subcontinua are arc-sets, we have at once the

COROLLARY. *A necessary and sufficient condition that every monotone transformation on A be monotone on every subcontinuum of A is that A be a dendrite.*

THEOREM 3.2. *A non-alternating transformation $T(A) = B$ is non-alternating on every arc-set in A .*

Let K be any arc-set in A , and suppose that T is not non-alternating on R . Then there exist points x and y of $T(K)$ such that $T^{-1}(x)$ separates R between some two points p and q of $T^{-1}(y)$. But since R is an arc-set, any set which separates two points in R must separate these points in A . Hence $T^{-1}(x)$ must separate p from q in A , which contradicts the hypothesis that T is non-alternating on A .

The following converse of this theorem is also true: if every non-alternating transformation $T(A) = B$ is non-alternating on a subcontinuum R of A , then R is an arc-set. In this statement, however, we can weaken the hypothesis by replacing the first occurrence of "non-alternating" by "completely non-alternating," without destroying the validity of the conclusion. From this follows, by virtue of Theorem 3.2, the equivalence of the apparently stronger and weaker forms of the hypothesis of the converse. Before proceeding to the proof of this converse, we establish two preliminary lemmas.

LEMMA A. *If a closed set M irreducibly separates a Peano space A between two points p and q , no proper subset of M separates M in A .*

Let C_p and C_q be the components of $A - M$ containing p and q respectively. Since M separates A irreducibly between p and q , M is the common boundary of C_p and C_q [8, p. 48, Th. 7], whence $M \subset \overline{C_p}$. Suppose that there exists a proper subset M' of M which separates M in A ; i. e., suppose that there exist mutually separated sets M'_1 and M'_2 such that $A - M' = M'_1 + M'_2 \neq 0$ and $M \cdot M'_1 \neq 0 \neq M \cdot M'_2$. Since $C_p \subset A - M \subset A - M'$, and since C_p is connected, C_p is contained either in M'_1 or in M'_2 , say in M'_1 . Hence $\overline{C_p} \subset \overline{M'_1}$, whence $M \subset \overline{M'_1}$. But this implies, since M'_1 and M'_2 are mutually separated, that $M \cdot M'_2 = 0$, contrary to our supposition.

LEMMA B. *If C is a continuum without cut points such that for some two points p and q of C there exist three mutually exclusive connected subsets of C , K_1 , K_2 , and K_3 , each having both p and q as limit points, and if p' and q' are points lying in different components of $C - p - q$, then $C - p' - q'$ is connected.*

Since C has no cut points, every component of $C - p' - q'$ must have both p' and q' as limit points. Hence, if C_i is a component of $C - p' - q'$ containing neither p nor q , $C_i + p' + q'$ is a connected set joining p' and q' in $C - p - q$. But this is impossible, for by hypothesis p' and q' lie in different components of $C - p - q$. Hence every component of $C - p' - q'$ contains either p or q . But $K_i \cdot (p' + q') = 0$ for at least one of the K_i ($i = 1, 2, 3$), and the connected set $K_i + p + q$ thus joins p and q in $C - p' - q'$, whence p and q lie in the same component of $C - p' - q'$. Hence $C - p' - q'$ has only one component, and thus is connected.

THEOREM 3.3. *If every completely non-alternating transformation $T(A) = B$ is non-alternating on a subcontinuum R of A , then R is an arc-set.*

Suppose R is not an arc-set. Then there exists an arc α , with end points p_1 and p_2 , such that $R \cdot \alpha = p_1 + p_2$. Thus p_1 and p_2 are joined by the continua α and R , having in common only $p_1 + p_2$, whence α lies in some true cyclic element C of A . As will be pointed out later, we shall lose no generality by assuming at this point that the space A is cyclicly connected, so that $C = A$.

Let K be a closed set which irreducibly separates [8, Th. 8] A between p_1 and p_2 . Since K and α are closed sets, $K \cdot \alpha$ is closed, whence, R being closed, $\rho(K \cdot \alpha, R) = d > 0$. Hence there exists an open set U (viz., the ϵ -neighborhood of $K \cdot \alpha$, where $\epsilon < d$) such that $U \supset K \cdot \alpha$ and $UR = 0$. Let $K' = K - U$.

Consider now the components C_i of $A - p_1 - p_2$ such that C_i contains a component C_i^* of $R - p_1 - p_2$ such that C_i^* has both p_1 and p_2 as limit

points. Since A is locally connected, there is only a finite number⁴ of such components C_i . We now consider separately Case 1, in which there is only one such component (that there is at least one is obvious), and Case 2, in which there is more than one.

Case 1. We define the transformation T as follows: $T(C_1 \cdot K) = z$, $T(p_1 + p_2) = t$, where z and t are distinct points, and T is homeomorphic elsewhere on A . Now T is not non-alternating on R , for $C_1 K'$ separates p_1 from p_2 in R (since $UR = 0$), whence $T^{-1}(z)$ separates $T^{-1}(t)$ in R . But T is completely non-alternating on A , for (1) no closed subset of $C_1 K'$ separates p_1 from p_2 in A (since U contains $K \cdot \alpha$, whence $K' \cdot \alpha = 0$), (2) no closed subset of $C_1 K'$ separates $C_1 K'$ in A (for any such separating subset of $C_1 K'$ would separate K in A , which is impossible by Lemma A), (3) no closed subset of $p_1 + p_2$ separates $C_1 K'$ in A (since $C_1 \supset C_1 K'$), (4) no closed subset of $p_1 + p_2$ separates $p_1 + p_2$, and (5) no single point can separate either $p_1 + p_2$ or $C_1 K'$ (since A was assumed to be cyclic).

Case 2. If there is more than one of the components C_i , each such component must intersect K' , whence $p_1 + p_2$ separates $K' \cdot \Sigma C_i$ in A , and a transformation defined as in Case 1 fails even to be non-alternating on A . To avoid this difficulty we redefine our transformation T as follows.

Let $K''_i = K' \cdot C_i$ and $K'' = \Sigma K''_i = K' \cdot \Sigma C_i$, noting that this is a finite sum. Then each K''_i is closed, whence K'' is closed, and therefore $\rho(p_1 + p_2, K'') = d > 0$. The space A being locally connected, there exists a positive number δ such that every point in $U_\delta(p_i)$ ⁵ can be joined to p_i by an arc lying within $U_d(p_i)$ ($i = 1, 2$). Let p'_i be any point of $U_\delta(p_i) \cdot C_i$, and let β_i be an arc $p_i p'_i$ lying within $U_d(p_i)$ ($i = 1, 2$). Now K'' separates p'_1 from p'_2 in R , for K'' separates p_1 from p_2 in R and $K'' \cdot U_d(p_i) = 0$ ($i = 1, 2$). Hence if we define T as the transformation which carries K'' into a point z and $p'_1 + p'_2$ into another point t , and is homeomorphic elsewhere on A , T is not non-alternating on R . But T is completely non-alternating on A , for (1) no closed subset of K'' separates p'_1 from p'_2 in A (since the connected set $\beta_1 + \alpha + \beta_2$ does not intersect K''), (2) no closed subset of K'' separates K'' in A (for the reason cited in (2) of Case 1), (3) no closed subset of $p'_1 + p'_2$ separates K'' in A (in fact, if in Lemma B we let $p = p_1$, $q = p_2$, $p' = p'_1$, $q' = p'_2$, $K_1 = C_1$, $K_2 = C_2$, and $K_3 = \alpha - p_1 - p_2$, we see that $p'_1 + p'_2$

⁴ For if the number of such components C_i is infinite, let $\{x_i\}$ be a sequence of points such that x_i is a point of C_i and $p_1 \neq x_i \neq p_2$ ($i = 1, 2, \dots$). Then, since A is compact, the sequence has a limit point x , and in the neighborhood of x the space fails to be locally connected.

⁵ The symbol $U_r(x)$ denotes the neighborhood of x of radius r .

does not cut A at all), (4) no closed subset of $p'_1 + p'_2$ separates $p'_1 + p'_2$, and (5) no single point separates $p'_1 + p'_2$ or K'' or itself in A (since A is assumed to be cyclic).

If the space were not assumed to be cyclicly connected, the foregoing steps would remain unaltered if T were defined to be a homeomorphism on $A - C$.

We have thus defined, on the supposition that R is not an arc-set, a transformation completely non-alternating on A but not non-alternating on R , contrary to hypothesis. Hence R is an arc-set.

THEOREM 3.4. *In order that every non-separating transformation $T(A) = B$ be non-separating on a given subcontinuum R of A , it is necessary and sufficient that R be an arc-set.*

The sufficiency is proved in the same way as Theorem 3.2. To prove the necessity, we shall make slight modifications in the proof of Theorem 3.3. Using the notation of that proof, and supposing as before that R is not an arc-set, consider the set $A - K'' = C_{p_1 p_2} + \Sigma C'_i$, where $C_{p_1 p_2}$ is the component of $A - K''$ containing p_1 and p_2 (these points lie in the same component of $A - K''$ because they are joined by the arc α , which does not intersect K'') and the C'_i are the other components of $A - K''$. Let $K''' = K'' + \Sigma C'_i$. Now all the limit points in $A - C'_i$ of any component C'_i must lie in K'' , whence $^{\circ}F(C'_i) \subset K''$ for all C'_i , and therefore $\Sigma F(C'_i) \subset K''$. But K'' is closed, so that $\overline{\Sigma F(C'_i)} \subset K''$, and, since the space is Peanian, $F(\Sigma C'_i) \subset \overline{\Sigma F(C'_i)} \subset K''$ [9, p. 155]. Therefore K''' is closed, and $A - K''' (= C_{p_1 p_2})$ is connected, whence K''' does not separate A . On the other hand, $\Sigma C'_i$ contains neither p_1 nor p_2 and K'' separates R between p_1 and p_2 (as we have seen in the proof of Theorem 3.3), so that K''' separates R between p_1 and p_2 . Therefore, assuming A to be cyclic, if T be defined to carry K''' into a point z and to be homeomorphic elsewhere on A , T is non-separating on A but not non-separating on R , contrary to hypothesis. If A is not cyclic and C is the true cyclic element of A containing $p_1 + p_2$, let T carry any component of $A - C$ into its limit point in C ; the result then follows as before. The supposition that R is not an arc-set thus leads to a contradiction and the theorem is proved.

Whether the theorem remains true if the first occurrence of "non-separating" be replaced by "completely non-separating," is not yet known.

Theorems 3.1-3.4 may be summarized in the

COROLLARY. *If A is a Peano space and R is a subcontinuum of A , the*

^{*} The symbol $F(x)$ denotes the boundary of the set X , i. e., the set $\overline{X} \cdot \overline{A - X}$.

following properties are equivalent: (1) R is an arc-set, (2) every transformation monotone on A is monotone on R , (3) every transformation non-alternating on A is non-alternating on R , (4) every transformation completely non-alternating on A is non-alternating on R , (5) every transformation non-separating on A is non-separating on R .

In the same way that Theorem 3.2 was established, we can prove

THEOREM 3.5. *If a transformation $T(A) = B$ is completely non-alternating (resp. completely non-separating, weakly non-alternating, weakly non-separating), it has the same property on every arc-set in A .*

Some of the converses of Theorem 3.4 are also true. Since Theorem 3.3 holds, it is true *a fortiori* that if, in that theorem, "non-alternating" or "weakly non-alternating" be substituted for "completely non-alternating," or if "completely non-alternating" be substituted for "non-alternating," or both, the theorem still holds. However, if "weakly non-alternating" be substituted for "non-alternating" in Theorem 3.3, then even if "weakly non-alternating" be substituted for "completely non-alternating" the resulting theorem is false. As indicated in the remark following Theorem 3.4, we do not know whether that theorem remains true if "completely non-separating" be substituted for "non-separating." In other respects the foregoing remarks concerning the various non-alternating transformations apply without change to the corresponding types of non-separating transformations.

4. Extension theorems. In this section we establish an extension theorem for monotone, non-alternating, and non-separating transformations; *i. e.*, a theorem which asserts that a transformation having one of these properties on certain subcontinua of the space has the same property on the whole space.

W. L. Ayres has defined [10, p. 568] a *basic set* of a Peano space A to be a subset of A such that each point of A lies on an arc joining two points of the subset. In this section we show that transformations which are monotone, non-alternating, or non-separating on all the cyclic chains joining pairs of points of a basic set possess their defining properties on the whole space.

THEOREM 4.1. *If K is a basic set of A , and if $T(A) = B$ is a transformation monotone on $C(x, y)$ for every pair of points x, y in K , then T is monotone on A .*

Suppose that T is not monotone on A ; then there exists a point b of B such that $T^{-1}(b) = U \cup V$, where U and V are non-vacuous mutually separated sets. Let u and v be points of U and V , respectively; then no connected subset

of $T^{-1}(b)$ contains $u + v$. We consider separately the cases in which (1) $u + v$ lies in some true cyclic element of A , and (2) u and v lie in distinct cyclic elements of A .

Case 1. If u and v lie in some true cyclic element C , let s be a non-cut point of A contained in C . Since K is a basic set of A , there exist points x and y in K such that $s \in C(x, y)$. But if an arc-set contains a non-cut point of a cyclic element, it contains the cyclic element.⁷ Hence $C(x, y) \supset C \supset u + v$. But then $T^{-1}(b)$ is not connected in $C(x, y)$ and T is not monotone on $C(x, y)$, contrary to hypothesis.

Case 2. If u and v do not lie in the same cyclic element, there exist cyclic elements C_u and C_v , containing u and v , respectively, and such that either $C_u \cdot C_v = 0$ or $C_u \cdot C_v$ is a single point. We proceed to find points x and y of K such that $C(x, y) \supset u + v$. If C_u is a node (*i. e.*, contains exactly one cut point of A), then [11, Th. 8] C_u contains a point of K which is not a cut point of A ; call this point x . On the other hand, if C_u is not a node (*i. e.*, contains more than one cut point of A), let c be the particular cut point in C_u which is the limit point of the component of $A - C_u$ containing C_v , and take as x any point of K lying in a component [10, Th. 11] M of $A - C_u$ whose limit point in C_u is a point d distinct from c .

Now we may select a point y in exactly the way that x was chosen, merely interchanging u and v in the above paragraph. Thus $C(x, y)$ contains either a non-cut point of C_i or two cut points of C_i ($i = u, v$), and in either case $C(x, y) \supset C_i$. Hence $C(x, y) \supset C_u + C_v \supset u + v$, whence T is not monotone on $C(x, y)$, contrary to hypothesis.

We now prove, in much simpler fashion, that the same theorem holds for non-alternating and non-separating transformations.

THEOREM 4.2. *If K is a basic set of A , and if $T(A) = B$ is a transformation non-alternating (non-separating) on $C(x, y)$ for every pair of points x, y in K , then T is non-alternating (resp. non-separating) on A .*

If T is not non-alternating on A , there exist points p and q of B such that $T^{-1}(p)$ separates A between some two points u and v of $T^{-1}(q)$, whence $T^{-1}(p)$ separates u from v in any connected subset of A containing $u + v$. But we have seen in the proof of Theorem 4.1 that there exist points x and y

⁷ Every cyclic chain is an arc-set, and it follows immediately from the definition of the latter that a non-degenerate arc-set contained in a cyclic element C must contain C ; and an arc-set which is not contained in C but contains a non-cut point in C must also contain a cut point in C , and hence, containing two points of C , must contain C .

of K such that $C(x, y) \supset u + v$. Hence $T^{-1}(p)$ separates $C(x, y)$ between u and v , contrary to the hypothesis that T is non-alternating on $C(x, y)$ for all points x, y in K . In the same way, T may be shown to be non-separating on A if it is non-separating on each $C(x, y)$.

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